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SEQUENTIALLY COHEN-MACAULAY MODULES AND LOCAL COHOMOLOGY

JÜRGEN HERZOG AND ENRICO SBARRA

Introduction

Let $I \subset R$ be a graded ideal in the polynomial ring $R = K[x_1, \dots, x_n]$ where K is a field, and fix a term order $<$. It has been shown in [17] that the Hilbert functions of the local cohomology modules of R/I are bounded by those of $R/\operatorname{in}(I)$, where $\operatorname{in}(I)$ denotes the initial ideal of I with respect to $<$. In this note we study the question when the local cohomology modules of R/I and $R/\operatorname{in}(I)$ have the same Hilbert function. A complete answer to this question can be given for the generic initial ideal $\operatorname{Gin}(I)$ of I, where $\operatorname{Gin}(I)$ is taken with respect to the reverse lexicographical order and where we assume that $\operatorname{char}(K) = 0$. In this case our main result (Theorem 3.1) says that the local cohomology modules of R/I and $R/\operatorname{Gin}(I)$ have the same Hilbert functions if and only if R/I is sequentially Cohen-Macaulay.

In Section 1 we give the definition of sequentially CM-modules which is due to Stanley [18], and in Theorem 1.4 we present Peskine's characterization of sequentially CM-modules in terms of Ext-groups. This characterization is used to derive a few basic properties of sequentially CM-modules which are needed for the proof of the main result.

In the following Section 2 we recall some well-known facts about generic initial modules, and also prove that $R/\operatorname{Gin}(I)$ is sequentially CM, see Theorem 2.2. Section 3 is devoted to the proof of the main theorem, and in the final Section 4 we state and prove a squarefree version (Theorem 4.1) of the main theorem. Its proof is completely different from that of the main theorem in the graded case. It is based upon a result on componentwise linear ideals shown in [2] and the fact (see [11]) that the Alexander dual of a squarefree componentwise linear ideal defines a sequentially CM simplicial complex.

1. Sequentially Cohen-Macaulay modules

We introduce sequentially Cohen-Macaulay modules and derive some of their basic properties. Throughout this section we assume that R is a standard graded Cohen-Macaulay K-algebra of dimension n with canonical module ω_R.

The following definition is due to Stanley [18, Section II, 3.9].

Definition 1.1. Let M be a finitely generated graded R-module. The module M is *sequentially Cohen-Macaulay* if there exists a finite filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots \subset M_r = M$$

of M by graded submodules of M such that each quotient M_i/M_{i-1} is CM, and $\dim(M_1/M_0) < \dim(M_2/M_1) < \ldots < \dim(M_r/M_{r-1})$.

The following observation follows immediately from the definition:

Lemma 1.2. (a) *Suppose that M is sequentially CM with filtration $0 = M_0 \subset M_1 \subset \ldots M_r = M$. Then for any $i = 0, \ldots, r$, the module M/M_i is sequentially CM with filtration $0 = M_i/M_i \subset M_{i+1}/M_i \subset \cdots \subset M_r/M_i$.*

(b) *Suppose that $M_1 \subset M$ and M_1 is CM, and M/M_1 is sequentially CM with $\dim M_1 < \dim M/M_1$. Then M is sequentially CM.*

In order to simplify notation we will write $E^i(M)$ for $\operatorname{Ext}^i_R(M, \omega_R)$.

Proposition 1.3. *Suppose that M is sequentially CM with a filtration as in* 1.1*, and assume that $d_i = \dim M_i/M_{i-1}$. Then*

(a) $E^{n-d_i}(M) \cong E^{n-d_i}(M_i/M_{i-1})$*, and is CM of dimension d_i for $i = 1, \ldots, r$, and $E^j(M) = 0$ if $j \notin \{n - d_1, \ldots, n - d_r\}$.*

(b) $E^{n-d_i}(E^{n-d_i}(M)) \cong M_i/M_{i-1}$ *for $i = 1, \ldots, r$.*

Proof. (a) We proceed by induction on r. From the short exact sequence $0 \to M_1 \to M \to M/M_1 \to 0$ we obtain the long exact sequence

$$\cdots \longrightarrow E^j(M/M_1) \longrightarrow E^j(M) \longrightarrow E^j(M_1) \to E^{j+1}(M/M_1) \longrightarrow \cdots$$

From [6, Theorem 3.3.10] it follows that $E^j(M_1) = 0$ if $j \neq n - d_1$, and that $E^{n-d_1}(M_1)$ is CM of dimension d_1. Thus we get an exact sequence

$$\begin{aligned} (1) \qquad 0 \ &\to E^{n-d_1}(M/M_1) \to E^{n-d_1}(M) \to E^{n-d_1}(M_1) \\ &\to E^{n-d_1+1}(M/M_1) \to E^{n-d_1+1}(M) \to 0, \end{aligned}$$

and isomorphisms $E^j(M/M_1) \cong E^j(M)$ for all $j \neq n - d_1, n - d_1 + 1$.

By Lemma 1.2, the module M/M_1 is sequentially CM and has a CM filtration of length $r - 1$. Hence by induction hypothesis we have $E^{n-j}(M/M_1) = 0$ for $j \neq \{d_2, \ldots, d_r\}$. This implies that $E^{n-d_1}(M/M_1) = E^{n-d_1+1}(M/M_1) = 0$, and hence by (1) we have $E^{n-d_1}(M) \cong E^{n-d_1}(M_1)$, and $E^{n-d_1+1}(M) = 0$. Summing up we conclude that $E^{n-d_1}(M) \cong E^{n-d_1}(M_1)$ and $E^j(M) \cong E^j(M/M_1)$ for $j \neq n - d_1$. Thus the assertion follows from the induction hypothesis and the fact that $E^{n-d_1}(M_1)$ is CM of dimension d_1.

(b) follows from (a) and [6, Theorem 3.3.10] since for any CM-module N of dimension d one has $N \cong E^{n-d}(E^{n-d}(N))$. $\square$

It is quite surprising that 1.3 has a strong converse. The following theorem is due to Peskine. Since there is no published proof available we present here a proof for the convenience of the reader.

Theorem 1.4. *The following two conditions are equivalent:*

(a) *M is sequentially CM;*

(b) *for all $0 \le i \le \dim M$, the modules $E^{n-i}(M)$ are either 0 or CM of dimension i.*

The implication (a) $\Rightarrow$ (b) follows from 1.3. For the other direction we first need to show

Lemma 1.5. *Let $t = \operatorname{depth} M$, and suppose that $E^{n-t}(M)$ is CM of dimension t. Then there exists a natural monomorphism $\alpha : E^{n-t}(E^{n-t}(M)) \to M$, and the induced map $E^{n-t}(\alpha)\colon E^{n-t}(M) \to E^{n-t}(E^{n-t}(E^{n-t}(M))) = E^{n-t}(M)$ is an isomorphism.*

Proof. We write $R = S/I$, where S is a polynomial ring. Let $\mathfrak{m}$ be the graded maximal ideal of R, and $\mathfrak{n}$ the graded maximal ideal of S. By the Local Duality Theorem (see [6, Theorem 3.6.10]) we have

$$\operatorname{Ext}^i_R(M, \omega_R) \cong \operatorname{Hom}_R(H^{n-i}_{\mathfrak{m}}(M), E_R(K)), \tag{2}$$

and

$$\operatorname{Ext}^i_S(M, \omega_S) \cong \operatorname{Hom}_S(H^{m-i}_{\mathfrak{n}}(M), E_S(K)), \tag{3}$$

where $m = \dim S$. Since $H^i_{\mathfrak{n}}(M) \cong H^i_{\mathfrak{m}}(M)$, and since $\operatorname{Hom}_S(R, E_S(K)) \cong E_R(K)$, we see that

$$\begin{aligned} \operatorname{Hom}_S(H^j_{\mathfrak{n}}(M), E_S(K)) &\cong \operatorname{Hom}_S(H^j_{\mathfrak{m}}(M), E_S(K)) \\ &\cong \operatorname{Hom}_R(H^j_{\mathfrak{m}}(M), \operatorname{Hom}_S(R, E_S(K)) \\ &\cong \operatorname{Hom}_R(H^j_{\mathfrak{m}}(M), E_R(K)). \end{aligned}$$

Therefore (2) and (3) imply that

$$\operatorname{Ext}^{n-t}_R(M, \omega_R) \cong \operatorname{Ext}^{m-t}_S(M, \omega_S),$$

and we hence may as well assume that R is a polynomial ring.

Let

$$F_\bullet : 0 \longrightarrow F_{n-t} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

be the minimal graded free resolution of M. Note that $\omega_R = R(-n)$ since R is a polynomial ring. Then the ω_R-dual $F^*_\bullet$ of $F_\bullet$ is the free complex

$$0 \longrightarrow F^*_{n-t} \longrightarrow \cdots \longrightarrow F^*_1 \longrightarrow F^*_0 \longrightarrow 0$$

with $F^*_i = \operatorname{Hom}_R(F_{n-i}, \omega_R)$, and $H_0(F^*) = E^{n-t}(M)$.

Let $G_\bullet$ be the minimal graded free resolution of $E^{n-t}(M)$. Then there exists a comparison map $\varphi_\bullet\colon F^*_\bullet \longrightarrow G_\bullet$ which extends the identity on $H_0(F^*_\bullet) = E^{n-t}(M) = H_0(G_\bullet)$.

Since by assumption $E^{n-t}(M)$ is CM of dimension t, the complex $G_\bullet$ has the same length as $F^*_\bullet$, namely $n-t$. Thus the ω_R-dual $\varphi^*_\bullet\colon G^*_\bullet \to F_\bullet$ of $\varphi_\bullet$ induces a natural homomorphism $\alpha = H_0(\varphi^*_\bullet)\colon E^{n-t}(E^{n-t}(M)) = H_0(G^*_\bullet) \to H_0(F_\bullet) = M$. Here $G^*_i = \operatorname{Hom}_R(G_{n-i}, \omega_R)$ and $\varphi^*_i = \operatorname{Hom}_R(\varphi_{n-i}, \omega_R)$ for all i.

Since $E^{n-t}(M)$ is CM by assumption, the complex $G^*_\bullet$ is exact, and hence a free resolution of $E^{n-t}(E^{n-t}(M))$, and so the induced map $E^{n-t}(E^{n-t}(E^{n-t}(M))) \to E^{n-t}(M)$ is given by $H_0(\varphi^{**}) = H_0(\varphi) = \mathrm{id}$.

It remains to be shown that α is a monomorphism. Let $C_\bullet$ be the mapping cone of $\varphi^*_\bullet :\ G^*_\bullet \to F_\bullet$. Since $F_\bullet$ and $G^*_\bullet$ are acyclic, it follows that $H_1(C_\bullet) \cong \operatorname{Ker}(\alpha)$ and $H_i(C_\bullet) = 0$ for $i > 1$. Notice that φ^*_{n-t} is an isomorphism, since this is the case for

φ_0. Hence the chain map $C_{n-t+1} \to C_{n-t}$ is split injective, and so by cancellation we get a new complex of free R-modules

$$\tilde{C}_\bullet : 0 \longrightarrow D_{n-t} \to C_{n-t-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0,$$

where $D_{n-t} = \mathrm{Coker}(C_{n-t+1} \to C_{n-t})$. Again we have $H_1(\tilde{C}_\bullet) \cong \mathrm{Ker}(\alpha)$ and $H_i(\tilde{C}_\bullet) = 0$ for $i > 1$.

Now suppose that $\mathrm{Ker}(\alpha) \neq 0$, and let P be a minimal prime ideal of the support of $\mathrm{Ker}(\alpha)$. Since $\mathrm{Ker}(\alpha) \subset E^{n-t}(E^{n-t}(M))$, and since $E^{n-t}(E^{n-t}(M))$ is a CM-module of dimension t, it follows that P is a minimal prime ideal of $E^{n-t}(E^{n-t}(M))$ with height $P = n-t$. Therefore $L_\bullet = \tilde{C}_\bullet \otimes R_P$, is a complex of length $n-t$ with $\operatorname{depth}(L_i) = n-t$ for all i, $\operatorname{depth}(H_1(L_\bullet)) = 0$ and $H_i(L_\bullet) = 0$ for $i > 0$. By the Peskine-Szpiro lemme d'acyclicité [16] this implies that $\tilde{C}$ is acyclic, a contradiction. □

Proof of 1.4. We proceed by induction on $n-t$. Let $t = \operatorname{depth} M$, and let M_1 be the image of $E^{n-t}(E^{n-t}(M)) \to M$. By 1.5, the module M_1 is a CM-module of dimension t.

Consider the short exact sequence $0 \to M_1 \to M \to M/M_1 \to 0$. As in the proof of 1.3 we get an exact sequence

$$\begin{aligned} 0 \; &\to E^{n-t}(M/M_1) \to E^{n-t}(M) \to E^{n-t}(M_1) \\ &\to E^{n-t+1}(M/M_1) \to E^{n-t+1}(M) \to 0, \end{aligned}$$

and isomorphisms $E^j(M/M_1) \cong E^j(M)$ for all $j \neq n-t, n-t+1$.
Since $E^{n-t}(M) \to E^{n-t}(M_1)$ is an isomorphism (see 1.5), we deduce from the above exact sequence that $E^{n-t}(M/M_1) = 0$, and that $E^{n-t+1}(M/M_1) \cong E^{n-t+1}(M)$. Thus we have $E^j(M/M_1) \cong E^j(M)$ for $j < n-t$, and $E^j(M/M_1) = 0$ for $j \geq n-t$. Hence, by induction hypothesis, M/M_1 is sequentially CM, and so is M by 1.2. □

An immediate application of 1.4 is

Corollary 1.6. *Let M be a finite direct sum of sequentially CM-modules. Then M is sequentially CM.*

As a consequence of 1.3 and 1.5 we get

Corollary 1.7. *A filtration of a sequentially CM module satisfying the conditions of* 1.1 *is uniquely determined.*

Proof. Let $t = \operatorname{depth} M$. The first module M_1 in the filtration must be the image of $E^{n-t}(E^{n-t}(M)) \to M$. Then one makes use of an induction argument to M/M_1 to obtain the desired result. □

Notice that $M_1 = H^0_{\mathfrak{m}}(M)$ if $\operatorname{depth} M = 0$. Thus, 1.1 together with 1.2 imply

Corollary 1.8. *An R-module M is sequentially CM if and only if $M/H^0_{\mathfrak{m}}(M)$ is sequentially CM.*

In what follows we denote by $E^\bullet(M) = \bigoplus_i E^i(M)$. Then we get

Corollary 1.9. *Suppose that $x \in R$ is a homogeneous M- and $E^\bullet(M)$-regular element. Then M is sequentially CM if and only if M/xM is sequentially CM.*

Proof. Since x is $E^\bullet(M)$-regular, the long exact Ext-sequence derived from

$$0 \longrightarrow M(-1) \xrightarrow{x} M \longrightarrow M/xM \to 0$$

splits into short exact sequences

$$0 \longrightarrow E^{n-i}(M) \xrightarrow{x} E^{n-i}(M)(1) \longrightarrow E^{n-i+1}(M/xM) \longrightarrow 0.$$

It follows that $E^{n-i}(M)$ is CM of dimension i if and only if $E^{n-i+1}(M/xM)$ is CM of dimension $i-1$. Thus 1.4 implies the assertion. □

In conclusion we would like to remark that the same theory is valid in the category of finitely generated R-modules, where R is a local CM ring and a factor ring of a regular local ring.

2. Generic initial modules

In this section we recall a few facts on generic initial modules, which are mostly due to Bayer and Stillman, and can be found in [7].

Let $R = K[x_1, \dots, x_n]$ be the polynomial ring over a field K of characteristic 0, and let M be a graded module with graded free presentation F/U. Throughout this section let $<$ be a term order that refines the partial order by degree and that satisfies $x_1 > x_2 > \cdots > x_n$. We fix a graded basis $e_1, \dots, e_m$ of F, and extend the order $<$ to F as follows: Let ue_i and ve_j be monomials (i.e. u and v are monomials in R). We set $ue_i > ve_j$ if either $\deg(ue_i) > \deg(ve_j)$, or the degrees are the same and $i < j$, or $i = j$ and $u > v$.

We set

$$U^{sat} = \bigcup_r U : \mathfrak{m}^r = \{f \in F \colon f\mathfrak{m}^r \in U \quad \text{for some } r\}.$$

From now on let $<$ denote the reverse lexicographic order. In the next proposition we collect all the results which will be needed later.

Proposition 2.1. *For generic choice of coordinates one has:*

(a) $\dim F/\operatorname{Gin}(U) = \dim F/U$ *and* $\operatorname{depth} F/\operatorname{Gin}(U) = \operatorname{depth} F/U$*;*
(b) x_n *is* F/U *regular if and only if* x_n *is* $F/\operatorname{Gin}(U)$ *regular;*
(c) $\operatorname{Gin}(U)^{sat} = \operatorname{Gin}(U^{sat})$.

Proof. After a generic choice of coordinates we may assume that $\operatorname{Gin}(U) = \operatorname{in}(U)$. The first statement in (a) is true for any term order, while the second statement about the depth and assertion (b) follow from [7, Theorem 15.13] because we may assume that the sequence $x_n, x_{n-1}, \dots, x_{n-t+1}$ is M-regular if $\operatorname{depth} M = t$.

By the module version of [7, Proposition 15.24], and by [7, Proposition 15.12] one has that

$$\operatorname{Gin}(U)^{sat} = \bigcup_r (\operatorname{Gin}(U) : x_n^r) = \bigcup_r \operatorname{Gin}(U : x_n^r).$$

On the other hand for a generic choice of coordinates we have $U^{sat} = \bigcup_r (U : x_n^r)$. Therefore $\bigcup_r \operatorname{Gin}(U : x_n^r) = \operatorname{Gin}(\bigcup_r (U : x_n^r)) = \operatorname{Gin}(U^{sat})$, which yields the last assertion. $\square$

The following result will be crucial for the proof of the main theorem of this paper.

Theorem 2.2. *The module $F/\operatorname{Gin}(U)$ is sequentially CM.*

Proof. Observe that, since we assume $\operatorname{char}(K) = 0$, we have $\operatorname{Gin}(U) = \bigoplus_j I_j e_j$ where for each j, I_j is a strongly stable ideal , cf. [7, Theorem 15.23]. Hence $F/\operatorname{Gin}(U) \cong \bigoplus_j R/I_j$, so that, by 1.6 one only has to prove that R/I is sequentially CM for any strongly stable ideal $I \subset R$.

Recall that a monomial ideal is strongly stable if for all $u \in G(I)$ and all i such that x_i divides u one has $x_j(u/x_i) \in I$ for all $j < i$. Here $G(I)$ denotes the unique minimal set of monomial generators of I.

For a monomial u we let $m(u) = \max\{i\colon x_i \text{ divides } u\}$, and $s = \max\{m(u)\colon u \in G(I)\}$. Let $R' = K[x_1, \dots, x_s]$, and let $J \subset R'$ be the unique monomial ideal such that $I = JR$. It is clear that J is a strongly stable ideal in R'. Thus it follows that $J^{sat} = \bigcup_r (J : x_s^r)$. Note that J^{sat} contains J properly and is strongly stable. Let $I_1 = \bigcup_r (I : x_s^r)$. Then $I_1 = J^{sat}R$, and since the extension $R' \to R$ is flat, we have $I_1/I \cong (J^{sat}/J) \otimes_{R'} R$. Now J^{sat}/J is a non-trivial 0-dimensional CM module over R', and therefore $M_1 = I_1/I \subset R/I$ is an $(n-s)$-dimensional CM-module over R. Next we observe that $(R/I)/M_1 = R/I_1$ and that I_1 is strongly stable. Since $\dim R/I_1 \geq n - \max\{m(u)\colon u \in G(I_1)\} > n - s$, the assertion of the theorem follows from 1.2. $\square$

3. The main theorem

As in the previous section we let K be a field of characteristic 0, $R = K[x_1, \dots, x_n]$ be the polynomial ring over K and M be a finitely generated graded R-module with graded free presentation $M = F/U$. We want to compare the Hilbert functions of the local cohomology modules of F/U and $F/\operatorname{Gin}(U)$, where $\operatorname{Gin}(U)$ is taken with respect to the reverse lexicographical order. In general one has (see [17]) a coefficientwise inequality $\operatorname{Hilb}(H^i_{\mathfrak{m}}(F/U)) \leq \operatorname{Hilb}(H^i_{\mathfrak{m}}(F/\operatorname{Gin}(U)))$. The main purpose of this section is to prove

Theorem 3.1. *The following conditions are equivalent:*

(a) *F/U is sequentially CM;*
(b) *for all $i \geq 0$ one has $\operatorname{Hilb}(H^i_{\mathfrak{m}}(F/U)) = \operatorname{Hilb}(H^i_{\mathfrak{m}}(F/\operatorname{Gin}(U)))$.*

Proof. (a) $\Rightarrow$ (b): Set $M = F/U$ and $N = F/\operatorname{Gin}(U)$. We proceed by induction on $\dim M$. Suppose $\dim M = 0$, then $\dim N = 0$ and $\operatorname{Hilb}(M) = \operatorname{Hilb}(N)$. Since $H^0_{\mathfrak{m}}(M) = M$, $H^0_{\mathfrak{m}}(N) = N$ and $H^i_{\mathfrak{m}}(M) = H^i_{\mathfrak{m}}(N) = 0$ for $i > 0$, the assertion follows in this case.

Now suppose that $\dim M > 0$. Assume first that $\operatorname{depth} M = 0$. We have $M/H^0_{\mathfrak{m}}(M) \cong F/U^{sat}$, and, by 2.1, $N/H^0_{\mathfrak{m}}(N) = F/\operatorname{Gin}(U)^{sat} = F/\operatorname{Gin}(U^{sat})$. By 1.8, we also know that $M/H^0_{\mathfrak{m}}(M)$ is sequentially CM. Thus, if the implication

(a) $\Rightarrow$ (b) were known for modules of positive depth, it would follow that

$$\mathrm{Hilb}(H^i_{\mathfrak m}(M)) = \mathrm{Hilb}(H^i_{\mathfrak m}(M/H^0_{\mathfrak m}(M))) = \mathrm{Hilb}(H^i_{\mathfrak m}(N/H^0_{\mathfrak m}(N))) = \mathrm{Hilb}(H^i_{\mathfrak m}(M))$$

for all $i > 0$. Notice that $H^0_{\mathfrak m}(M) = U^{sat}/U$ and $H^0_{\mathfrak m}(N) = \mathrm{Gin}(U^{sat})/\mathrm{Gin}(U)$. However, since $M = F/U$ and $N = F/\mathrm{Gin}(U)$, and since F/U^{sat} and $F/\mathrm{Gin}(U^{sat})$ have the same Hilbert function, we conclude that also $H^0_{\mathfrak m}(M)$ and $H^0_{\mathfrak m}(N)$ have the same Hilbert function.

These considerations show that we may assume that $\operatorname{depth} M > 0$. Accordingly, $\operatorname{depth} N > 0$ by 2.1, and N is sequentially CM by 2.2. Since M and N are sequentially CM we have $\operatorname{depth} E^\bullet(M) > 0$ and $\operatorname{depth} E^\bullet(N) > 0$. We may assume that the coordinates are chosen generically so that $\mathrm{Gin}(U) = \mathrm{in}(U)$ and that x_n regular on $E^\bullet(M)$ and regular on $E^\bullet(N)$. According to 1.9, $M/x_nM = F/(U + x_nF)$ is sequentially CM. Therefore our induction hypothesis, together with [7, Proposition 15.12], implies that the Hilbert functions of the local cohomology modules of M/x_nM and of $F/\mathrm{Gin}(U + x_nF) = F/(\mathrm{Gin}(U) + x_nF) = N/x_nN$ are the same.

We have short exact sequences of graded R-modules

$$0 \longrightarrow H^{i-1}_{\mathfrak m}(M/x_nM) \longrightarrow H^i_{\mathfrak m}(M)(-1) \xrightarrow{x_n} H^i_{\mathfrak m}(M) \longrightarrow 0,$$

and

$$0 \longrightarrow H^{i-1}_{\mathfrak m}(N/x_nN) \longrightarrow H^i_{\mathfrak m}(N)(-1) \xrightarrow{x_n} H^i_{\mathfrak m}(N) \longrightarrow 0,$$

because x_n is regular on $E^\bullet(M)$ and $E^\bullet(N)$. Therefore applying the induction hypothesis to M/x_nM we get

$$\begin{aligned}\mathrm{Hilb}(H^i_{\mathfrak m}(M))(t-1) &= \mathrm{Hilb}(H^{i-1}_{\mathfrak m}(M/x_nM)) \\ &= \mathrm{Hilb}(H^{i-1}_{\mathfrak m}(N/x_nN)) = \mathrm{Hilb}(H^i_{\mathfrak m}(N))(t-1),\end{aligned}$$

from which we deduce that $\mathrm{Hilb}(H^i_{\mathfrak m}(M)) = \mathrm{Hilb}(H^i_{\mathfrak m}(N))$.

(b) $\Rightarrow$ (a): We proceed again by induction on $\dim M$. With the same arguments as in the proof of the first implication, we may assume that $\operatorname{depth} M > 0$. Therefore $\operatorname{depth} N > 0$, too, and since we are working with generic coordinates and N is sequentially CM by 2.2, x_n is $E^\bullet(N)$-, M- and N-regular. We shall show that x_n is also $E^\bullet(M)$-regular. Since x_n is $E^\bullet(N)$-regular, the long exact cohomology sequence derived from $0 \to N(-1) \overset{x_n}{\to} N \to N/x_nN \to 0$ splits into short exact sequences

$$0 \longrightarrow H^{i-1}_{\mathfrak m}(N/x_nN) \longrightarrow H^i_{\mathfrak m}(N)(-1) \xrightarrow{x_n} H^i_{\mathfrak m}(N) \longrightarrow 0, \tag{4}$$

We show by induction on i that the corresponding sequences for M are also exact. For $i = 0$ the assertion is trivial, since $H^0_{\mathfrak m}(M) = 0$. Now let $i > 0$, and assume that the assertion is true for all $j < i$. Then $H^{i-1}_{\mathfrak m}(M)(-1) \overset{x_n}{\to} H^{i-1}_{\mathfrak m}(M)$ is surjective, and we obtain the exact sequence

$$0 \longrightarrow H^{i-1}_{\mathfrak m}(M/x_nM) \longrightarrow H^i_{\mathfrak m}(M)(-1) \xrightarrow{x_n} H^i_{\mathfrak m}(M).$$

Suppose multiplication with x_n is not surjective, then there exists a degree a such that $H^i_{\mathfrak m}(M)_{a-1} \xrightarrow{x_n} H^i_{\mathfrak m}(M)_a$ is not surjective. Using (4), and the hypothesis that

the local cohomology modules of M and N have the same Hilbert function, one has

$$\begin{aligned}\dim_K H^{i-1}_{\mathfrak{m}}(M/x_nM)_a &> \dim_K H^i_{\mathfrak{m}}(M)_{a-1} - \dim H^i_{\mathfrak{m}}(M)_a \\ &= \dim_K H^i_{\mathfrak{m}}(N)_{a-1} - \dim H^i_{\mathfrak{m}}(N)_a \\ &= \dim_K H^{i-1}_{\mathfrak{m}}(N/x_nN)_a.\end{aligned}$$

This is a contradiction, since $M/x_nM = F/(U + x_nF)$ and $N/x_nN = F/\operatorname{Gin}(U + x_nF)$, and consequently $\dim_K H^{i-1}_{\mathfrak{m}}(M/x_nM) \leq \dim_K H^{i-1}_{\mathfrak{m}}(N/x_nN)$.

Now it follows that x_n is $E^\bullet(M)$-regular, and also that the cohomology modules of M/x_nM and N/x_nN have the same Hilbert functions, whence our induction hypothesis implies that M/x_nM is sequentially CM. Since x_n is $E^\bullet(M)$-regular, we finally deduce that M is sequentially CM. □

4. The squarefree case

In this section we will state and prove the squarefree analogue of Theorem 3.1. Let Δ be a simplicial complex on the vertex set $[n] = \{1, \dots, n\}$, and let $I_\Delta \subset R$ be the Stanley-Reisner ideal of Δ, where $R = K[x_1, \dots, x_n]$ and K is field of characteristic 0. The K-algebra $K[\Delta] = R/I_\Delta$ is the Stanley-Reisner ring of Δ.

We recall the concept of symmetric algebraic shifting which was introduced by Kalai in [15]: Let $u \in R$ be a monomial, $u = x_{i_1}x_{i_2}\cdots x_{i_d}$ with $i_1 \leq i_2 \leq \ldots \leq i_d$. We define

$$u^\sigma = x_{i_1}x_{i_2+1}\cdots x_{i_d+d-1}.$$

Note that u^σ is a squarefree monomial (in a possibly bigger polynomial ring).

As usual the unique minimal monomial set of generators of a monomial ideal I is denoted by $G(I)$.

The *symmetric algebraic shifted complex of* Δ is the simplicial complex Δ^s whose Stanley-Reisner ideal I_{Δ^s} is generated by the squarefree monomials u^σ with $u \in \operatorname{Gin}(I_\Delta)$.

We quote the following properties of I_{Δ^s} from [1] and [2]:

(i) I_{Δ^s} is a strongly stable ideal in R;
(ii) one has the following inequality of graded Betti numbers:

$$\beta_{ij}(I_\Delta) \leq \beta_{ij}(I_{\Delta^s});$$

(iii) I_Δ and I_{Δ^s} have the same graded Betti numbers if and only if I_Δ is componentwise linear.

Recall that an ideal $I \subset R$ is called *componentwise linear* if in each degree i, the ideal generated by the i-th graded component I_i of I has a linear resolution.

Let Δ^* denote the Alexander dual of Δ, i.e., the simplicial complex

$$\Delta^* = \{F \subset [n] \colon [n] \setminus F \notin \Delta\}.$$

It has been noted in [11, Theorem 9] that

$$K[\Delta] \quad \text{is sequentially CM} \Longleftrightarrow I_{\Delta^*} \quad \text{is componentwise linear.}$$

Theorem 4.1. *Let Δ be a simplicial complex. Then*

(a) $\mathrm{Hilb}(H^i_{\mathfrak{m}}(K[\Delta])) \leq \mathrm{Hilb}(H^i_{\mathfrak{m}}(K[\Delta^s]))$ *for all* i.
(b) *The local cohomology module of* $K[\Delta]$ *and* $K[\Delta^s]$ *have the same Hilbert function if and only if* $K[\Delta]$ *is sequentially CM.*

Proof. Part (a) is proved in [17]. For the proof of (b) we shall need the following result which also can be found in [17]: for all $i \geq 0$ and $j \geq 0$ one has

$$\dim_K H^i_{\mathfrak{m}}(K[\Delta])_{-j} = \sum_{h=0}^{n} \binom{n}{h}\binom{h+j-1}{j}\beta_{i-h+1,n-h}(K[\Delta^*]). \tag{5}$$

(Observe that $H^i_{\mathfrak{m}}(K[\Delta])_j = 0$ for $j > 0$ and all i, as shown by Hochster, see [13] and [6, Theorem 5.3.8]).

Now suppose that $K[\Delta]$ is sequentially CM. Then I_{Δ^*} is componentwise linear, and hence $\beta_{ij}(K[\Delta^*]) = \beta_{ij}(K[(\Delta^*)^s])$ by Property (iii) of symmetric algebraic shifting. Since $(\Delta^*)^s = (\Delta^s)^*$, Formula (5) shows that $\dim_K H^i_{\mathfrak{m}}(K[\Delta])_{-j} = \dim_K H^i_{\mathfrak{m}}(K[\Delta^s])_{-j}$ for all i and j, as desired.

For the viceversa, let H be the $(n+1)\times(n+1)$-matrix with entries $h_{ij} = \dim_K H^i_{\mathfrak{m}}(K[\Delta])_{-j}$, $i,j = 0,\dots,n$, B the $(n+1)\times(n+1)$-matrix with entries $b_{hi} = \binom{n}{h}\beta_{i-h+1,n-h}(K[\Delta^*])$ and A the $(n+1)\times(n+1)$-matrix with entries $a_{jh} = \binom{h+j-1}{j}$. Then (5) says that $H^t = AB$. Since A is invertible, we see that the numbers $b_{hi} = \binom{n}{h}\beta_{i-h+1,n-h}$ are determined by the Hilbert functions of the local cohomology modules of $K[\Delta]$. Thus, if $\dim_K H^i_{\mathfrak{m}}(K[\Delta])_j = \dim_K H^i_{\mathfrak{m}}(K[\Delta^s])_j$ for all i and j, then the numbers b_{hi} for Δ^* and $(\Delta^*)^s$ coincide, which in turn implies that their graded Betti numbers are the same (because $\beta_{ij} = b_{n-j,i+n-j-1}/\binom{n}{n-j}$). By Property (iii) of symmetric algebraic shifting this implies that I^*_Δ is componentwise linear, and hence $K[\Delta]$ is sequentially CM. $\square$

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Jürgen Herzog, Fachbereich Mathematik und Informatik, Universität-GHS Essen, 45117 Essen, Germany

E-mail address: `juergen.herzog@@uni-essen.de`

Enrico Sbarra, Fachbereich Mathematik und Informatik, Universität-GHS Essen, 45117 Essen, Germany

E-mail address: `enrico.sbarra@@uni-essen.de`